

I. (10 pts) Express the repeating decimal $a = 0.42121212\dots = 0.4\overline{21}$ as the ratio of two integers.

$$a = 0.4212121\dots = 0.4 + 0.02121\dots$$

$$= \frac{4}{10} + \frac{1}{10} (0.2121\cancel{2}\dots)$$

$$= \frac{4}{10} + \frac{1}{10} \left(\frac{21}{100} + \frac{21}{100^2} + \dots \right)$$

$$= \frac{4}{10} + \frac{1}{10} \cdot \frac{21}{100} \left(1 + \frac{1}{100} + \frac{1}{100^2} + \dots \right)$$

$$= \frac{4}{10} + \frac{21}{1000} \cdot \sum_{n=1}^{\infty} \frac{1}{100^{n-1}}$$

$$= \frac{4}{10} + \frac{21}{1000} \cdot \left(\frac{1}{1 - \frac{1}{100}} \right)$$

$$= \frac{4}{10} + \frac{21}{1000} \cdot \frac{100}{99} = \frac{4}{10} + \frac{21}{990} = \frac{4}{10} + \frac{7}{330}$$

$$= \frac{132+7}{330} = \frac{139}{330} \quad \left(= \frac{417}{990} \right)$$

II. (15 pts) Consider the series $S = \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$.

a. Prove that S converges.

b. Find two real numbers a and b such that $\frac{2n+1}{n^2(n+1)^2} = \frac{a}{n^2} - \frac{b}{(n+1)^2}$.

c. Deduce the value of the series S .

a) ~~$\frac{2n+1}{n^2(n+1)^2}$~~ at ∞ $\sim \frac{2}{n^3}$, but $\sum \frac{2}{n^3} \text{ CV}$.

then by the LCT $\sum \frac{2n+1}{n^2(n+1)^2} \text{ CV}$.

$$b) \frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

$$\begin{aligned} c) S_n &= \frac{1}{1^2} - \frac{1}{2^2} \\ &\quad + \frac{1}{2^2} - \frac{1}{3^2} \\ &\quad + \frac{1}{3^2} - \frac{1}{4^2} \\ &\quad \vdots \\ &\quad \vdots \\ &\quad + \frac{1}{n^2} - \frac{1}{(n+1)^2} \\ \hline &= 1 - \frac{1}{(n+1)^2} \end{aligned}$$

$$\sum \frac{2n+1}{n^2(n+1)^2} = \lim_{n \rightarrow \infty} S_n = 1.$$

III. (20 pts) Find x such that the following geometric series converge

$$a. \sum_{n=1}^{\infty} \frac{3^n}{(2x-1)^n}, \quad b. \sum_{n=1}^{\infty} \left(\frac{1}{x}-1\right)^n, \quad c. \sum_{n=0}^{\infty} \frac{1}{\cosh^n(x)}, \quad d. \sum_{n=1}^{\infty} \tan^n(x).$$

$$a) -1 < \frac{3}{2x-1} < 1 \quad \text{then}$$

$$\frac{2x-1}{3} < -1 \quad \text{or} \quad \frac{2x-1}{3} > 1.$$

$$x < -1 \quad \text{or} \quad x > 2.$$

$$x \in]-\infty, -1] \cup [2, \infty[$$

$$b) -1 < \frac{1}{x}-1 < 1 \quad (\Rightarrow) \quad 0 < \frac{1}{x} < 2 \quad (\Rightarrow) \quad x \in]\frac{1}{2}, \infty[$$

$$c) -1 < \frac{1}{\cosh(x)} < 1 \quad (\Rightarrow) \quad \forall x \neq 0 \quad (\Rightarrow) \quad x \in \mathbb{R}^*$$

$$d) -1 < \tan(x) < 1 \quad x \in]-\frac{\pi}{4}, \frac{\pi}{4}[\cup [\frac{3\pi}{4}, \frac{5\pi}{4}[$$

$$+ (\kappa\pi) \cdot$$

IV. (25 pts) Study the convergence of the following series

a. $\sum_{n=1}^{\infty} \frac{3^n + 5^n}{4^n + 7^n}$ b. $\sum_{n=1}^{\infty} \left(1 - \frac{2}{n}\right)^{\frac{n^2}{2}}$ c. $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$ d. $\sum_{n=1}^{\infty} \left(1 + \frac{\sqrt{2}}{n}\right)^{-n^2}$ e. $\sum_{n=1}^{\infty} \frac{(n!)^2}{2^{n^2}}$

a.) $\sum \frac{3^n + 5^n}{4^n + 7^n}$ $\frac{3^n + 5^n}{4^n + 7^n} \approx \left(\frac{5}{7}\right)^n$ but $\sum \left(\frac{5}{7}\right)^n$ cv

then by the LCT $\sum \frac{3^n + 5^n}{4^n + 7^n}$ cv .

b.) $\sum \left(1 - \frac{2}{n}\right)^{\frac{n^2}{2}}$
 Root test: $\left(1 - \frac{2}{n}\right)^{\frac{n^2}{2}} = \left(1 - \frac{2}{\frac{n^2}{2}}\right)^{\frac{n^2}{2}} \xrightarrow[n \rightarrow \infty]{} e^{-1} < 1$
 so $\sum \left(1 - \frac{2}{n}\right)^{\frac{n^2}{2}}$ cv .

c.) $\sum \frac{2^n n!}{n^n}$ Radio test: $\frac{u_{n+1}}{u_n}$
 $\frac{u_{n+1}}{u_n} = \frac{2^{n+1} (n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n n!} = 2 \cdot \frac{n^n}{(n+1)^n} = 2 \left(\frac{n}{n+1}\right)^n \xrightarrow[n \rightarrow \infty]{} 2e^{-1} < 1$ so it converges.

d.) $\sum \left(1 + \frac{\sqrt{2}}{n}\right)^{-n^2}$ Root test
 $\left(1 + \frac{\sqrt{2}}{n}\right)^{-n} = \frac{1}{\left(1 + \frac{\sqrt{2}}{n}\right)^n} \xrightarrow[n \rightarrow \infty]{} \frac{1}{e^{\sqrt{2}}} < 1$
 so it converges.

e. $\sum \frac{(n!)^2}{2^n}$. Ratio test.

$$\frac{u_{n+1}}{u_n} = \frac{2^{(n+1)^2}}{2^{n^2}} \cdot \frac{2^n}{(n!)^2} = (n+1)^2 \cdot 2^{-2n-1}$$

$$= \frac{(n+1)^2}{2^{2n+1}} \xrightarrow[n \rightarrow \infty]{} 0 < 1 \text{ so it converges.}$$

V. (15 pts) Study the conditional convergence of the following series

$$a. \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n} \quad b. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{e^n} \quad c. \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{1+n}$$

$$a) \sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln(n)}$$

$$\left\{ \begin{array}{l} \frac{1}{n \ln(n)} > 0 \text{ for } n \geq 2 \\ \frac{1}{n \ln(n)} \xrightarrow[n \rightarrow \infty]{} 0 \\ \frac{1}{(n+1) \ln(n+1)} < \frac{1}{n \ln n} \text{ so it } \downarrow. \end{array} \right.$$

then by Leibniz' theorem this series CV.

let's study the absolute convergence

$\sum \frac{1}{n \ln(n)}$, the conditions for the \int test are satisfied

$$\text{then } \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \sim \int_2^{\infty} \frac{dx}{x \ln(x)} = \lim_{b \rightarrow \infty} [\ln \ln(x)]_2^b = \infty$$

so this series is conditionally convergent.

$$b) \sum (-1)^{n+1} \frac{n}{e^n} \quad \left\{ \begin{array}{l} \frac{n}{e^n} > 0 \text{ for } n \geq 1 \\ \lim_{n \rightarrow \infty} \frac{n}{e^n} = 0 \\ \frac{n}{e^n} \downarrow \end{array} \right. \quad \left\{ \begin{array}{l} f(x) = \frac{x}{e^x} \quad f'(x) = \frac{e^x(1-x)}{e^{2x}} \leq 0 \text{ for } x > 1 \\ \text{so } \frac{n}{e^n} \downarrow \end{array} \right.$$

then by Leibniz this series converges.

let's study the abs. conv: $\sum \frac{n}{e^n} \geq \sum n e^{-n}$ the conditions for \int test are satisfied then $\sum \frac{n}{e^n} \sim \int x e^{-x} dx$

$$\int x e^{-x} dx = \lim_{b \rightarrow \infty} [-x e^{-x} - e^{-x}] \Big|_0^b$$

x	e^{-x}
1	$-e^{-x}$
0	e^{-x}

$$= \lim_{b \rightarrow \infty} (-b e^{-b} - e^{-b} + e^0 + e^0) = 0 - \frac{2}{e} \quad (\text{CV})$$

then this series is absolutely convergent so it's not conditionally convergent.

$$\therefore \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{1+n}$$

Absolute CV: $\sum \frac{\sqrt{n}}{1+n}$; $\frac{\sqrt{n}}{1+n} \underset{\text{at } \infty}{\sim} \frac{1}{\sqrt{n}}$
so it does not CV absolutely

Now if we return to the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{1+n}$$

Leibniz: $\left\{ \begin{array}{l} \frac{\sqrt{n}}{1+n} > 0 \\ \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{1+n} = 0 \\ \frac{\sqrt{n}}{1+n} \downarrow \end{array} \right.$

$$\begin{aligned} f(x) &= \frac{\sqrt{x}}{1+x} \\ f'(x) &= \frac{1}{2\sqrt{x}(1+x)} - \frac{\sqrt{x}}{(1+x)^2} \\ &= \frac{1-x}{2\sqrt{x}(1+x)^2} < 0 \quad \forall x > 0 \\ \text{then } f'(x) &\downarrow \end{aligned}$$

so it's conditionally CV.

VI. (10 pts) Let a be a ~~real~~^{positive} number. Study, in function of a , the behavior of the series

$$\sum_{n=1}^{\infty} \left(\frac{n+a}{n+1} \right)^{n^2}$$

Root test: $\left(\frac{n+a}{n+1} \right)^n = \left(1 + \frac{a-1}{n+1} \right)^n$

$$= \left(1 + \frac{a-1}{n+1} \right)^{n+1} \left(1 + \frac{a-1}{n+1} \right)^{-1}$$

$$\xrightarrow[n \rightarrow \infty]{} e^{a-1}$$

$$\begin{cases} a < 1 & e^{a-1} < 1 \quad CV. \\ a > 1 & e^{a-1} > 1 \quad DV \\ a = 1 & \text{the series becomes } \sum 1 \text{ which is DV.} \end{cases}$$

VII. (5 pts) Let $\sum a_n$ be a convergent series with $a_n > 0, \forall n > N$. Prove that $\sum \frac{\sin^2(a_n)}{a_n}$ converges. (Hint: recall that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$).

$\sum a_n$ cv. let $b_n = \frac{\sin^2(a_n)}{a_n} \geq 0$

$$\frac{b_n}{a_n} = \frac{\sin^2(a_n)}{a_n^2} \quad \lim_{n \rightarrow \infty} \frac{b_n}{a_n} = \lim_{n \rightarrow \infty} \frac{\sin^2(a_n)}{a_n^2} = 1.$$

because $\lim_{n \rightarrow \infty} a_n = 0$

then by the LCT $\sum \frac{\sin^2(a_n)}{a_n}$ behaves like $\sum a_n$
hence it converges

VIII. (10 pts) Let a and b be two digits and consider x the repeating number of the form $x = 0.ababab\dots$

a. Prove that $x = \frac{10a+b}{99}$

b. Use a. to write the number 2.2131313 as a ratio of two integers.

a. $x = 0.abab\dots = \frac{[ab]}{100} + \frac{[ab]}{100^2} + \dots$

with $[ab] = 10a+b$, then

$$\begin{aligned} x &= \frac{10a+b}{100} \left(1 + \frac{1}{100} + \frac{1}{100^2} + \dots \right) \\ &= \frac{10a+b}{100} \cdot \sum_{n=1}^{\infty} \frac{1}{100^{n-1}} \\ &= \frac{10a+b}{100} \cdot \frac{1}{1 - \frac{1}{100}} = \frac{10a+b}{99} \end{aligned}$$

b. $\Leftrightarrow 2.21313 = \frac{1}{10} \cdot 22.1313\dots$

$$= \frac{1}{10} (22 + 0.1313\dots)$$

now we apply a. on $0.1313\dots = \frac{13}{99} = \frac{13}{99}$

then $2.21313\dots = \frac{1}{10} \left(22 + \frac{13}{99} \right) = \frac{2191}{990}$